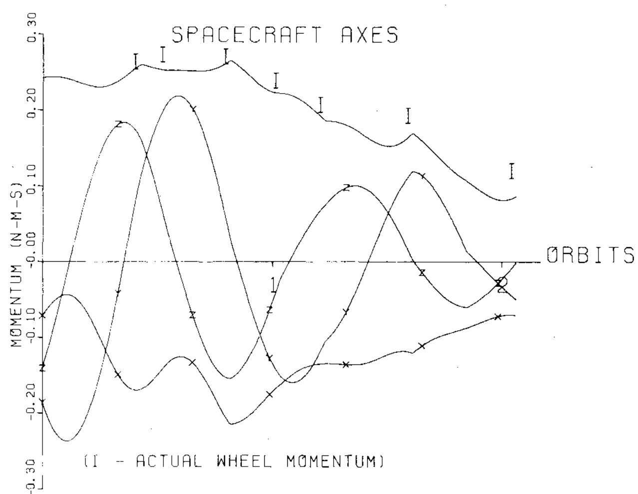

Fig. 5 Actual and predicted momentum vector components and magnitude—Aug. 8, 1979 test.

maximum dump because not all optimum switch times were schedulable. In fact, the predicted total momentum actually increased at four times, with these increases being caused by later than optimum switch times.

Conclusions

The main unknowns before testing were the usability of the Mead-Fairfield model at radii lower than its design regime and how much momentum could actually be dumped by on-orbit satellites using the magnetic control technique. It was found that the magnetic momentum dump algorithm works well with on-orbit GPS satellites using the Mead-Fairfield magnetic field model. As of Nov. 1, 1979, a total of 15 tests had been run with the four satellites in orbit at that time. The method has succeeded in dumping up to 0.33 N-m-s excess angular momentum in two orbits. For the specific satellite/orbit geometry combination tested, this corresponded to a buildup occuring over about 12 days.

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Special Cubic Solution Function

Dunstan Graham*
Princeton University, Princeton, N.J.

Introduction

CONSIDER the equation with real coefficients

$$ax^3+bx^2+cx+d=0 \qquad (a>0) \tag{1}$$

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*Consultant. Associate Fellow AIAA.

Girolamo Cardan[1] published a solution, but the general procedure is so tedious that it is seldom attempted. Instead, usually, recourse is had to iterative calculations. These may be greatly facilitated by either graphs or tables, and there have been several attempts to provide such aids in a convenient form (see, e.g., Refs. 2-6).

Although today there are very powerful methods for the numerical solution of polynomial equations (including cubic equations) and, typically, several such root solving methods are a part of the "libraries" associated with stored-program computers, an exclusive reliance on such methods may not always be advised. It is primarily to the development of a simple, general, and reliable method for the numerical solution of cubic equations on a hand-held calculator that this paper is addressed. Of course, if the cubic equation can be solved, the quartic equation is also amenable to solution by Ferrari's, Lagrange's, or a similar technique.[6]

Combining the methods of the reduced[1,2,6] and normalized equations[3,4] leads to a new and more convenient equivalent equation. A solution of this equation (corresponding to an isolated real root) has overlapping asymptotic properties, and easily learned asymptotic starting estimates can be used in effective procedures for very rapidly refining an approximate solution. The remaining two roots may then be found readily as the solutions to a depressed equation.[7] Charts and tables are superfluous.

The Normalized, Reduced Equation

In Eq. (1), set $x=y-b/3a$ This yields the reduced equation[7]

$$y^3+\left(\frac{c}{a}-\frac{b^2}{3a^2}\right)y+\left(\frac{d}{a}+\frac{2b^3}{27a^3}-\frac{bc}{3a^2}\right)=0 \tag{2}$$

Then set[4]

$$y=s\,|m|^{1/3}$$

where

$$m=\left(\frac{d}{a}+\frac{2b^3}{27a^3}-\frac{bc}{3a^2}\right)$$

This yields the normalized reduced equation

$$s^3+\frac{(c/a-b^2/3a^2)}{|m|^{2/3}}s+(1)\,\mathrm{sgn}(m)=0 \tag{3}$$

For the moment, *assume* $\mathrm{sgn}\,m>0$, so that

$$s^3+ks+1=0 \tag{4}$$

where

$$k=\frac{-3(b^2-3ac)}{|27a^2d+2b^3-9abc|^{2/3}}$$

and (since $a>0$), in fact, the sign of the trailing term in Eq. (4) will be the sign of $(27a^2d+2b^3-9abc)$.

Equation (4) will always have at least one real root s_1. This root, of course, is a function G of the parameter k and is opposite in sign to the quantity m. Thus,

$$s_1=-\mathrm{sgn}(27a^2d+2b^3-9abc)\,G(k) \tag{5}$$

Using the previous definitions, together with the understanding that $(\cdot)^{1/3}$ means the real cube root, the relationship between the root s_1 and a corresponding real root x_1 of Eq. (1) shows that

$$x_1=\frac{-b}{3a}+y_1=\frac{-b}{3a}-\frac{(27a^2d+2b^3-9abc)^{1/3}}{3a}G(k) \tag{6}$$

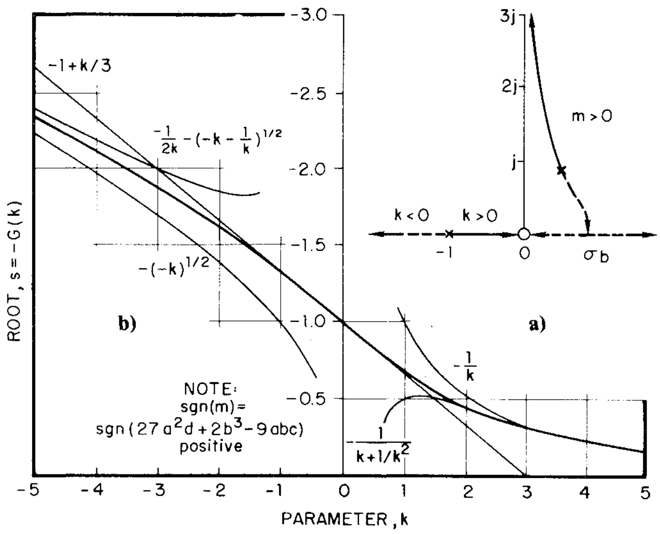

Fig. 1 The special cubic solution function: a) root locus; b) the negative real root and asymptotic approximations.

There may be an apparent difficulty in finding the root s_1 when $(27a^2d+2b^3-9abc)=0$. Observe, however, that in this special case, $y_1=0$ is a root of Eq. (2) and, as is known,[6] $x_1=-b/3a$ is a root of Eq. (1).

A solution s_1 of Eq. (4) as a function of the parameter k might be called the special cubic solution function. For the case in which the trailing coefficient is positive, a conventional top-half root locus[8] is shown in Fig. 1a. The root locus gain is the parameter k. The locus of the always real, *negative* root in question s_1 is shown in heavy lines. (These are solid when $k>0$, dashed when $k<0$.) When $(b^2-3ac)=0$, $k=0$ and the root $s_1=-1$. [If $(b^3-27a^2d)=b^2-3ac=0$, $x_{1,2,3}=-b/3a$ is a triple root of Eq. (1).[6]] Loci of the other roots, complex conjugates or positive reals, are also shown. The rendezvous point (double root) is at $\sigma_b=(1/2)^{1/3}$; there, the gain $k=-(27/4)^{1/3}$ and $s_b=-(4)^{1/3}$. Note that the *negative* real root is always isolated. In the case in which $m<0$, the root locus is geometrically the mirror image of the one shown. Previously negative roots are positive, and positive roots (or roots with positive real parts) are negative (or have negative real parts).

Still assuming $m>0$, the root $s_1=-G(k)$ is illustrated in Fig. 1b as the heavily lined curve. This solution may be calculated by showing zero remainder upon synthetic division of the left-hand side of Eq. (4) by a trial root. Recall that in the event m, or $(27a^2d+2b^3-9abc)<0$, the root s_1 has the same absolute value but is positive.

Asymptotic Properties

If a root is very small in absolute magnitude, the third-degree term in Eq. (4) is negligible compared to the others. Then, $s_i \cong -1/k$. But if this were the case, synthetic division[7] of the polynomial (left-hand side) of Eq. (4) would appear as

$$\begin{array}{cccc|l} 1 & 0 & k & 1 & \underline{-1/k} \\ & -1/k & +1/k^2 & -(1/k)(k+1/k^2) & \\ \hline 1 & -1/k & k+1/k^2 & R & \end{array} \tag{7}$$

where R is the (nonzero) remainder. At the same time, the conditions for $s_{i+1}=-1/(k+1/k^2)$ to be a valid next estimate for the root would be satisfied. Observe that this estimate is formed by a process called "cross-divide," as the negative quotient of the trailing coefficient and the term under the solidus in the third column. Both the approximate solutions s_i and s_{i+1} are asymptotic to $s=-G(k)$. Represented in Fig. 1 as light lines, they may be seen to be very accurate as long as $k \gg 1$.

Somewhat similarly, the geometry of the Jahnke and Emde nomogram[3] and the σ-Bode diagram[5,8] suggests $s_i \cong -\sqrt{-k}$ when $k \ll -1$. But when that is the case, there is a small positive root $\cong -1/k$. The depressed equation which Eq. (7) yields may then be used to form an expression for the negative root. Neglecting small terms in $1/k^2$ leads to $s_{i+1} \cong -(1/2k)-\sqrt{-k-(1/k)}$. Both these asymptotic approximations, when $k \ll -1$, are displayed in Fig. 1b.

Furthermore, the geometry of the Jahnke and Emde nomogram suggests $s_i \cong -1+k/3$ when $|k|<1$. In terms of the present redefinition of the parameters, this approximate solution is the same as the first two terms in the series developed by Neumark.[6] The line $s_i=-1+k/3$ is also shown on Fig. 1b.

Iterative Procedures

Now that asymptotic estimates are in hand, we may consider improving a starting estimate.

It has already been noted that when $k \gg 1$, a new root estimate may be formed by the cross-divide process. Repeated application of this procedure results in refinement of the root estimate. The Lipschitz condition[9] may be used to demonstrate that for $k>1$, the iterative procedure converges. Because of the simple nature of Eq. (4), synthetic division and cross-divide are particularly easy to carry out. The new estimate is explicit in the expression $s_{j+1}=-1/(k+s_j^2)$. Not only is this the case, but also on successive synthetic divisions, the remainder R will change sign. Linear interpolation, conveniently, might then be used between any two successive root estimates.

In the event that $k \le 1$, the cross-divide process is not appropriate, but Newton's method for successive approximations is advantageous.[10] In this case, a new approximation to a real root of Eq. (4) is explicit in the expression

$$s_{j+1}=s_j-\frac{R(s_j)}{\mathrm{d}R(s_j)/\mathrm{d}s_j}=s_j-\frac{(1+ks_j+s_j^3)}{k+3s_j^2} \tag{8}$$

Since the remainder $R(s_j)=1+ks_j+s_j^3$ from the previous trial is available, this calculation is also easy to carry out. And because the root we seek is always isolated, no great labor is involved in achieving remarkable accuracy. This is true even when the starting estimate (such as may be given by the asymptotic approximations) is crude.

Other Roots

While Eq. (4) has three roots as, of course, does Eq. (1) and, so far, only one has been found, the remaining two may be found from a depressed equation whose coefficients are available from the synthetic division process.[7] Since, however, the original problem was the solution of Eq. (1), it appears most convenient to now proceed directly to its solution. When the remainder, upon synthetic division of the polynomial left-hand side of Eq. (4) by a trial root s_n is sufficiently small,

$$x_n=\frac{-b}{3a}+\frac{(27a^2d+2b^3-9abc)^{1/3}}{3a}s_n \tag{9}$$

is the corresponding root of Eq. (1). Then, when synthetic division of the left-hand side of Eq. (1) appears, in part, as

$$\begin{array}{cccc|l} a & b & c & d & \underline{x_n} \\ & ax_n & x_n(b+ax_n) & \cdots & \\ \hline a & b+ax_n & c+x_n(b+ax_n) & \text{small remainder} & \end{array} \tag{10}$$

the depressed equation is

$$ax^2+(b+ax_n)s+(c+x_n(b+ax_n))=0$$

This depressed equation may be solved for the other two roots of Eq. (1) by means of the quadratic formula.

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Geometric Dilution of Precision in Global Positioning System Navigation

Bertrand T. Fang*
Computer Sciences Corporation, Silver Spring, Md.

Introduction

THE NAVSTAR global positioning system (GPS), when fully operational in the early 1990's, will provide worldwide navigation through synchronized transmissions from a constellation of eighteen 12-h period satellites in three 55-deg-inclination orbital planes. An accurate user navigation fix (position and time) will be obtainable by receiving transmissions from four satellites and decoding the signal transit times.

One may relate the measurements, referred to as the pseudoranges, to the navigation state as follows:

$$CT_j = \sqrt{(X_1 - x_1)^2 + (X_2 - x_2)^2 + (X_3 - x_3)^2} + X_4 + n_j \quad (1)$$

where C is the velocity of light; T_j is the signal transit time from GPS satellite j to user (not corrected for user clock offset Δt); X_1, X_2, X_3, X_4 are user navigation states, where the first three represent a set of convenient cartesian user coordinates, and $X_4 = C\Delta t$ is a range bias equivalent of user clock offset; x_1, x_2, x_3 are the corresponding cartesian coordinates of GPS satellite j; and n_j is random measurement noise.

From a set of four measurements, a user navigation fix may be determined. The accuracy of the fix is characterized by the following 4×4 position-time navigation error covariance matrix:

$$P = (H^T W H)^{-1} \quad (2)$$

where

$$H^T = \left[\begin{array}{c|c|c|c} a & b & c & d \\ \hline 1 & 1 & 1 & 1 \end{array}\right] \quad (3)$$

is the measurement partial derivative matrix, transposed; a, b, c, and d are line-of-sight unit vectors from a set of four GPS satellites to the user; W is the 4×4 covariance matrix of random measurement noise; and superscript T indicates the transpose of matrix.

The measurement error covariance matrix W is generally taken to be diagonal, which is strictly true for uncorrelated measurements only. In practice, assignments of quantitative values to the elements of W also take into consideration such factors as the elevation and health status of individual GPS satellites. Thus, W may more appropriately be referred to as the weighting matrix. For uniform weighting, P is proportional to $(H^T H)^{-1}$, which depends only on the relative geometry of the user and the four GPS satellites, as is evident from Eq. (3). The square root of the trace of $(H^T H)^{-1}$ is referred to as Geometric Dilution of Precision (GDOP), a self-explanatory name. Whatever the weighting strategy, the trace of the navigation error covariance matrix serves as a convenient and natural performance index characterizing the accuracy of the navigation fix. For a diagonal weighting matrix W, the trace of P is the sum of diagonal terms of $(H^T H)^{-1}$ weighted by the inverses of the corresponding elements of W.

Thus, the evaluation of the GPS navigation performance is essentially equivalent to the computation of the diagonal terms of $(H^T H)^{-1}$, which may be called the GDOP matrix for covenience.

The navigation performance index, $\mathrm{tr}(P)$, also serves as a criterion for the selection of a set of four best GPS satellites among those visible, which may be as many as ten for users which are satellites themselves. If, for optimum performance, each of the different combinations of four has to be evaluated, the computational burden can be considerable. In the following, certain theoretical results concerning the general properties of the GDOP matrix are derived. An efficient algorithm for the computation of the GDOP matrix and the navigation performance index is given; applications of the results are illustrated by numerical examples.

Analytical Results

To solve for a navigation fix from four measurements, the partial derivative matrix H must be nonsingular. Since the determination of H

$$|H| = \left|\begin{array}{c|c|c|c} a-d & b-d & c-d & d \\ \hline 0 & 0 & 0 & 1 \end{array}\right| = |a-d \,|\, b-d \,|\, c-d|$$

a navigation fix can be determined from four GPS satellites with line-of-sight directions a, b, c, and d if and only if the three vectors $(a-d)$, $(b-d)$, and $(c-d)$ are linearly independent, i.e., noncoplanar.

From Eq. (3) and the fact that a, b, c, and d are unit vectors, one obtains

$$\mathrm{tr}(H^T H)^{-1} = \mathrm{tr}(HH^T)^{-1}$$

$$= \mathrm{tr}\begin{bmatrix} 2 & a^T b+1 & a^T c+1 & a^T d+1 \\ & 2 & b^T c+1 & b^T d+1 \\ & & 2 & c^T d+1 \\ \text{Symmetric} & & & 2 \end{bmatrix}^{-1} \quad (4)$$

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*Senior Principal Engineer, Orbit Operations, System Sciences Division.